The Mixmaster Universe: the final reckoning?

2008 J. Phys. A: Math. Theor. 41 155201

(http://iopscience.iop.org/1751-8121/41/15/155201)

J. Phys. A: Math. Theor. **41** (2008) 155201 (11pp) doi:10.1088/1751-8113/41/15/155201

The Mixmaster Universe: the final reckoning?

K Andriopoulos and P G L Leach[1]

Department of Information and Communication Systems Engineering, University of the Aegean, Karlovassi 83 200, Greece

E-mail: kand@aegean.gr, leach@math.aegean.gr and leachp@ukzn.ac.za

Received 1 December 2007, in final form 23 February 2008
Published 2 April 2008
Online at stacks.iop.org/JPhysA/41/155201

Abstract
The investigation of the Mixmaster Universe using singularity analysis has attracted much attention in recent years and produced a variety of interpretations, some of which have been supported by numerical experimentation. We present a new singularity analysis and make direct comparisons with systems of known dynamical behaviour to avoid making unwarranted inferences and remove the controversy surrounding the Mixmaster Universe.

PACS numbers: 02.30.Ik, 02.40.Xx, 98.80.Jk

1. Introduction

The generic, nonrotating, homogeneous cosmological model for a closed space, i.e. one of Bianchi Type IX, commonly known as the Mixmaster Universe due to its complex behaviour, has been an object of considerable study since it was introduced independently by Belinskii and Khalatnikov [8, 9], Belinskii *et al* [10, 11] and Misner [33–35]. Various approaches were taken including that of Belinskii–Khalatnikov–Lifshitz, the Hamiltonian methods developed by Misner, the methods of dynamical systems [12, 38] and numerical experimentation [13, 14, 27]. The results of these studies produced a picture which could be at best described as unclear. The theoretical studies supported the view that the Mixmaster Universe was a purely chaotic dynamical system with its evolution backwards in time being unpredictable, indeed stochastic and erratic. The numerical evidence is not so decisive. Hobill *et al* [27] suggest that the behaviour in time of the Mixmaster Universe has a very sensitive dependence upon initial conditions. For some initial conditions the behaviour is chaotic while for other initial conditions the behaviour is quite regular. Christiansen *et al* [14] lead evidence that indicates that the Mixmaster Universe is 'very probably not integrable', which is not quite

[1] Permanent address: School of Mathematical Sciences, University of KwaZulu-Natal, Private Bag X54001 Durban 4000, South Africa.

the same thing as being chaotic. In particular the calculated Lyapunov exponent vanishes[2]. In a detailed numerical investigation of the Hamiltonian version of the system, Bountis and Drossos [13] provided numerical evidence of the nonintegrability of the Mixmaster model due to the patterns of singularities in the complex t-plane, which appear to accumulate densely. Cornish and Levin [19] make a numerical investigation of the Mixmaster Universe using the lower-dimensional Farey map to isolate regions in which the evolution of the system is chaotic[3]. In the same vein Creighton and Hobill [21] observe that there exists an infinite number of discrete self-similar solutions of basal period three. This is very suggestive of the period doubling associated with chaos and is reminiscent of the picture of chimneys of singularities described by Bountis and Drossos.

In the early Nineties of the last century there were several papers devoted to the singularity analysis of the system of ordinary differential equations describing the evolution of the system. In 1993 Contopoulos *et al* [16] reported that the equations of motion pass the Painlevé Test and this evidence for integrability was supported by an analysis based upon Ziglin's theorem [39, 40]. They were not successful in finding the two additional constants of motion required to demonstrate integrability in terms of the theorem of Liouville[4]. In their application of the ARS algorithm [1–3] Contopoulos *et al* distinguished two main cases. For one case there was generic singular behaviour with six free parameters. For the second there were only four free parameters but no incompatibilities. Consequently they concluded that the system of equations describing the Mixmaster Universe satisfied the Painlevé criterion for integrability. In a subsequent paper [17] the same authors revisited the Mixmaster Universe in the light of then recent developments in the interpretation of negative resonances. Kruskal [29] had observed that a negative resonance may induce multivaluedness and that the Painlevé–Laurent expansion can be considered as the lowest-order term of a perturbative series. The idea of a perturbative treatment of negative resonances was further developed by Fordy and Pickering [25] and Conte *et al* [15]. The application of the perturbative Painlevé approach to the system of equations for the Mixmaster Universe gave a negative result in terms of integrability. The implications of having one case indicating integrability and the other not were, and still are, understandably not obvious. In 1995 Contopoulos *et al* [18] reviewed the singularity analysis of the Mixmaster Universe and concluded that it possesses critical essential singularities and hence cannot be integrable. The presence of the negative resonance in the results of the singularity analysis was further examined and the authors proposed that the presence of a positive and a negative resonance led to the introduction of logarithmic terms due to their 'interaction'[5]. Their conclusion is that, on the basis of the information available at the time, the equations describing the Mixmaster Universe are neither integrable nor chaotic in the usual understanding of the term, but exhibit the transient chaos found in some scattering systems. The results reported by Contopoulos *et al* were compatible with those of Latifi *et al* [30] who showed that the perturbation of an exact solution exhibits a movable essential singularity and so general nonintegrability follows. In both papers it is shown that the Mixmaster system is not globally integrable. Integrability in some regions of the complex time–plane is not excluded. This is found to be the case for the three known closed-form solutions. In 1998 Springael *et al* [37] confirmed the existence of a four-parameter analytic solution—an extension to the three-parameter solution of Belinskii *et al* [11]—by means of the perturbative Painlevé Test.

[2] The distinction between a chaotic system and a nonintegrable system is important. Indeed even an integrable system such as the time-dependent linear oscillator can display quite wild evolution in time even when the nature of the time dependence is a simple periodic function [31].

[3] It is not completely clear whether they distinguish between nonintegrable and chaotic solutions.

[4] The so-called Hamiltonian of the system would provide the first of the required three integrals.

[5] The apostrophes are due to Contopoulos *et al*. The nature of the interaction was not specified.

At a more general level Demaret and Scheen [22] performed the perturbative analysis of the perfect fluid Bianchi Type IX relativistic cosmological model with a view to determine the existence of probable chaotic regimes. The system which they analyze coincides with the model of the Mixmaster Universe studied here in the case that the ratio of specific heats is 2, i.e. the presence of stiff matter. For general values of the ratio of specific heats their system fails the singularity analysis and exhibits infinitely many movable logarithmic singularities. They also detect single-valued particular solutions.

In 1994[6] Cotsakis and Leach [20] presented the first singularity analysis of the model of the Mixmaster Universe. They found only one of the cases reported subsequently by Contopoulos *et al* [16]. This was the case for which there are only four arbitrary constants in the solution, which indicates a particular or subsidiary solution. The form of the equations analyzed by Cotsakis and Leach is

$$\begin{aligned} u\ddot{u} - \dot{u}^2 &= u^2(v-w)^2 - u^4 \\ v\ddot{v} - \dot{v}^2 &= v^2(w-u)^2 - v^4 \\ w\ddot{w} - \dot{w}^2 &= w^2(u-v)^2 - w^4 \end{aligned} \tag{1}$$

and are subject to the constraint of

$$\dot{u}\dot{v}w + u\dot{v}\dot{w} + \dot{u}v\dot{w} - uvw(u^2+v^2+w^2-uv-vw-wu) = 0. \tag{2}$$

It is this form of the equations describing the Mixmaster Universe which we use in this paper. We note that in the analyses of Contopoulos *et al* a quasi-hamiltonian formulation is used. As our three dependent variables are the same as their position coordinates, a direct comparison of the results of the singularity analysis in each paper for these variables presents no problem. Neither Contopoulos *et al* [16] (indeed nor subsequently) nor Cotsakis and Leach [20] took the constraint, (2), into account. As Christiansen *et al* [14] remind us, the left-hand side of (2) is a first integral of the system (1), and consequently, the only effect of the constraint on the value of the integral being put to zero is that instead of six arbitrary constants in the solution of system (1) there are only five arbitrary constants. In other words the constraint does not cause a problem to the analysis [6].

A critical point in the singularity analysis of the system of differential equations modelling the Mixmaster Universe lies in the understanding of the results of the analysis. For example in [16] the less satisfactory result of the analysis is that there is a triple -1 resonance. This is attributed to forcing the singularities for three variables to be at the same point. Kruskal [29] interprets negative resonances as a requirement for the development of a perturbative series. However, another interpretation has been provided. Already in 1993 Lemmer and Leach [32] had demonstrated a Laurent expansion for the solution of the equation

$$\ddot{x} + 3x\dot{x} + x^3 = 0 \tag{3}$$

which commenced at the leading-order term and descended to minus infinity, which was a consequence of the existence of the resonance, -2, in addition to the so-called generic resonance, -1. A more considered development of the idea can be found in the paper of Feix *et al* [24] in which the concept of Left and Right Painlevé Series was introduced. The Right Painlevé Series was the one normally associated with the singularity analysis in that it was a Laurent series commencing at the singular term and going to the plus infinity. The Left Painlevé Series run from the singular term to minus infinity. These two series accounted for differential equations having positive or negative resonances. The problem—of immediate

[6] By one of those curious quirks associated with learned journals this paper, received on the fourth of February, 1993, did not appear until 1994 whereas the first paper of Contopoulos *et al* [16] appeared in June, 1993, after being submitted on the twelfth of March, 1993.

importance to the analysis of the differential equations describing the Mixmaster Universe—of differential equations possessing both positive and negative resonances (apart from -1) was resolved by way of a very simple example by Andriopoulos and Leach [4]. It is with this new knowledge that one can now return to the Mixmaster Universe to obtain a better understanding of its behaviour in terms of singularity analysis. In the process we see that there is a possible explanation of the difficulties encountered by Contopoulos *et al* [17]. We should emphasize that our interest in this paper is not the resolution of the integrability or nonintegrability of the differential equations describing the Mixmaster Universe but rather a proper understanding of the singularity analysis of the system.

This paper is constructed as follows. In section 2 (next section) we examine system (1) for its leading-order behaviour. This has relevance not only to the papers cited already when dealing with this subject but also to a more general interpretation of relevance to Cosmology which includes particular solutions. In section 3 we proceed to the formal singularity analysis of system (1) following the method established by Kowalevskaya [28] in her successful search for a new solution of the problem of a rigid body rotating about a fixed point. Some of the ambiguities of the application of the singularity analysis were resolved by a less sophisticated approach which considered the next-to-leading-order behaviour and further [24, 26], and we discuss these in section 4 not only in the context of the Mixmaster Universe but also for an equation which is explicitly integrable so that a direct comparison may be made. This is pertinent to the discussion of the presence of logarithmic terms in the expansion. In section 5 we present our concluding comments.

2. Leading-order behaviour

To determine the leading-order behaviour of system (1) we make the substitutions $u = \alpha\tau^p$, $v = \beta\tau^q$ and $w = \gamma\tau^r$, where $\tau = t - t_0$ and t_0 is the location of the putative singularity. From the balancing of exponents we obtain three possible cases, namely

case 1: $p = -1$, $q > -1$ and $r > -1$ et cyc,
case 2: $p = -1$, $q = -1$ and $r > -1$ et cyc,
case 3: $p = q = r = -1$.

All cases are compatible with the constraint as far as the exponents are concerned.

If we are looking for solutions analytically apart from isolated polelike singularities, the unspecified exponents in cases 1 and 2 must be integral. A first possibility for case 1 is that $q = r = 0$. Then the leading-order terms in (1) are, respectively, $\alpha^2 + \alpha^4$, $\alpha^2\beta^2$ and $\alpha^2\gamma^2$. These are not compatible with the definition of leading-order behaviour in the standard singularity analysis. For case 2 the leading-order terms are, respectively, $\alpha^2 + \alpha^4 - \alpha^2\beta^2$, $\beta^2 + \beta^4 - \alpha^2\beta^2$ and $(\alpha - \beta)^2\gamma^2$. Again these are not compatible with the requirement that all coefficients of the leading-order terms be nonzero. Thus we conclude that none of the exponents p, q and r can ever be zero.

Since the possibility of the exponents of the leading-order terms being zero has been excluded, we may look at system (1) in the standard approach of singularity analysis by taking the inverses of the functions with positive exponents for the leading-order terms. If we consider case 1 as it is written, we make the substitutions $w(t) \longrightarrow 1/y(t)$ and $v(t) \longrightarrow 1/z(t)$ into (1) to obtain the system

$$\begin{aligned}
&-u^2y^2 + 2u^2yz - u^2z^2 + u^4y^2z^2 - y^2z^2u'^2 + uy^2z^2u'' = 0 \\
&y^2 - z^2 + 2uyz^2 - u^2y^2z^2 + y^2z'^2 - y^2zz'' = 0 \\
&-y^2 + 2uy^2z + z^2 - u^2y^2z^2 + z^2y'^2 - yz^2y'' = 0.
\end{aligned} \tag{4}$$

When we make the substitutions $u = \alpha\tau^{-1}$, $z = \beta\tau^q$ and $y = \gamma\tau^r$, the equations to be solved for the coefficients of the leading-order terms are

$$\alpha^2\beta^2\gamma^2 + \alpha^4\beta^2\gamma^2 = 0 \qquad \beta^2\gamma^2 q - \alpha^2\beta^2\gamma^2 = 0 \qquad \beta^2\gamma^2 r - \alpha^2\beta^2\gamma^2 = 0.$$

The only acceptable solution is $q = r = -1$ with $1 + \alpha^2 = 0$, and β and γ being arbitrary. For case 2 the procedure is the same except that v is not transformed. The only possible value that r can take brings us to case 3.

We conclude that only case 1 and case 3 are possible candidates for the standard singularity analysis. When we solve for the coefficients of the leading-order terms for the latter case, we obtain $\alpha = \pm\mathrm{i}$, $\beta = \pm\mathrm{i}$ and $\gamma = \pm\mathrm{i}$ as the only possible nonzero values. Note that the integrable axially symmetric model, for which $v = w$, say, has the same two cases. Therefore we do not treat it separately.

The reported case [37] of $(0, -2, -2)$, indeed any of $(0, n, n)$, $n < -1$, n being an integer, presents some difficulties. The next-to-leading-order behaviour proves to exist for arbitrary functions of τ. There is no determining system for the resonances. The singularity analysis gives no result. Springael *et al* state that the $(0, -2, -2)$ case is found in [16, 17, 30], but we have been unable to discover it in a series of careful readings of the manuscripts.

So far we have been considering the application of the singularity analysis in its standard form which is that all coefficients of the leading-order terms be nonzero. One source of particular solutions can be found by allowing one or more of the coefficients of the leading-order terms to be zero. For case 3 for the coefficients of the leading-order terms we obtain the possibilities for (α, β, γ) to be $(\pm\mathrm{i}, 0, 0)$ *et cyc* in addition to the trivial solution and the one utilized above. There are no other particular solutions found.

3. The resonances

For case 1[7] we make the substitutions $u = \alpha\tau^{-1} + \mu\tau^{-1+s}$, $z = \beta\tau^{-1} + \nu\tau^{-1+s}$ and $y = \gamma\tau^{-1} + \xi\tau^{-1+s}$ into the dominant terms of system (4). We recall that $\alpha = \pm\mathrm{i}$ and β, γ are arbitrary. The terms linear in μ, ν and ξ give the system

$$\begin{bmatrix} (s^2 - s - 2) & 0 & 0 \\ -2\mathrm{i}b^2c^2 & bc^2(s^2 - s) & 0 \\ -2\mathrm{i}b^2c^2 & 0 & (s^2 - s)b^2c \end{bmatrix} \begin{bmatrix} \mu \\ \nu \\ \xi \end{bmatrix} = 0 \tag{5}$$

which is nontrivial for $s = -1, 0(2), 1(2), 2$. In [16, 17] the resonances are given as $-1, 0(4), 2$. Our values are in agreement with those given in [22, 37]. The corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ \mathrm{i}b \\ \mathrm{i}b \end{pmatrix} k_0, \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} k_1 \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} k_2, \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} k_3 \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} k_4, \qquad \begin{pmatrix} 1 \\ \mathrm{i}b \\ \mathrm{i}b \end{pmatrix} k_5,$$

respectively. No incompatibilities arise at the positive resonances. When the expansions up to the resonance $s = 2$ are substituted into the full system, no inconsistency arises.

We now consider case 3. The terms at which arbitrary constants enter the solution are obtained by means of the substitutions $u = \pm\mathrm{i}\tau^{-1} + \mu\tau^{-1+s}$, $v = \pm\mathrm{i}\tau^{-1} + \nu\tau^{-1+s}$ and $w = \pm\mathrm{i}\tau^{-1} + \xi\tau^{-1+s}$ into system (1) and extraction of the terms linear in μ, ν and ξ. It is well-known [16, 20] that the solution is $s = -1(3), 2(3)$. The triple root of 2 does not pose a problem as the geometric multiplicity of the eigenvectors is also three and three arbitrary constants enter the Laurent series in each of u, v and w at τ^1. The triple root of -1 did create

[7] We consider just the pattern indicated above. Obviously the same discussion applies cyclically.

something of a problem. Contopoulos *et al* [16] observed that no logarithmic terms entered at this resonance and attributed the triple root to the forcing of the singularity in each function to be at the same point[8]. Cotsakis and Leach [20] did not express an opinion. In both papers a solution in the form of Laurent series commencing at τ^{-1} containing four parameters—the arbitrary location of the movable singularity and the three constants entering at the positive resonance—was proposed. For Contopoulos *et al* this was simply a subsidiary solution and, as they already had a solution corresponding to $p = -1$, $q = 1$ and $r = 1$ (in our notation) containing six parameters, that was sufficient for them to conclude that the system of equations describing the Mixmaster Universe satisfied the Painlevé criterion for integrability. Cotsakis and Leach had to be content with just a four-parameter solution, i.e. a particular or subsidiary solution. This suggested that the Mixmaster Universe was integrable on a hyperline in the six-dimensional space of initial conditions[9], which becomes five-dimensional because of the constraint, (2).

4. Next-to-leading-order behaviour

In an attempt to understand the workings of the ARS algorithm rather than accepting it as a recipe to be followed, Marc Feix proposed that one should consider a general function instead of the specific monomial, $\mu\tau^{p+r}$, of the algorithm by setting

$$x = \alpha\tau^{p} + g(\tau) \tag{6}$$

after the leading-order behaviour had been determined. Evidently $g(\tau)$ is subdominant to the leading term. The determination of g gives an indication of the next-to-leading-order behaviour and it is easy to envisage a succession of such determinations. It is instructive to examine how the process works with an equation of known solution. An informative example is found in the third element of the Riccati Sequence [5, 23] which has proven to be very useful in interpreting the role played by resonances in the singularity analysis [4]. The equation is

$$\dddot{x} + 4x\ddot{x} + 3\dot{x}^{2} + 6x^{2}\dot{x} + x^{4} = 0. \tag{7}$$

There are three possibilities for the leading-order behaviour, namely τ^{-1}, $2\tau^{-1}$ and $3\tau^{-1}$. It is a simple matter to show that the resonances are $(-1, 1, 2)$, $(-2, -1, 0)$ and $(-3, -2, -1)$, respectively. The first and third cases correspond to a Right Painlevé Series and a Left Painlevé Series, respectively.

When we substitute (6) into (7) (with the appropriate values of α and p) we obtain a nonlinear equation for $g(\tau)$. In the case of a Right Painlevé Series $\tau g(\tau) \longrightarrow 0$ as $\tau \longrightarrow 0$ and similarly for the derivatives. We use this property to linearize the equation and obtain

$$\dddot{g} + 4\tau^{-1}\ddot{g} = 0$$

with solution

$$g = c_1\tau^{-2} + c_2 + c_3\tau,$$

[8] Curiously there appeared to be no necessity for the common location of the singularity to give rise to a triple resonance of -1 in case 1.

[9] One need not go to a system as complex as the Mixmaster Universe to find an example of a system integrable on a submanifold of the space of initial conditions. The equation $y''' + y'' + yy' = 0$, which has its origins in General Relativity, has a first integral and for a particular value of that first integral the equation is integrable. Otherwise it is not integrable. Indeed the solution is badly behaved albeit not chaotic [36].

where the c_i, $i = 1, 3$, are the constants of integration. The first term must be discarded since it is more singular than the first term of the Right Painlevé Series[10]. We can continue the process now by writing x in terms of the obtained partial solution and a new $g(\tau)$ as

$$x = \tau^{-1} + c_2 + c_3\tau + g(\tau).$$

The equation to be solved is

$$\dddot{g} + 4\tau^{-1}\ddot{g} + \left(4c_2^3 + 12c_2c_3\right)\tau^{-1} + c_2^4 + 18c_2^2c_3 + 15c_3^2 + \left(4c_2^3c_3 + 24c_2c_3^2\right)\tau \\ + \left(6c_2^2c_3^2 + 10c_3^3\right)\tau^2 + 4c_2c_3^3\tau^3 + c_3^4\tau^4 = 0$$

and it has the solution

$$g = c_4\tau^{-2} + c_5 + c_6\tau - \left(\tfrac{1}{2}c_2^3 + \tfrac{3}{2}c_2c_3\right)\tau^2 - \left(\tfrac{1}{30}c_2^4 + \tfrac{3}{5}c_2^2c_3 + \tfrac{1}{2}c_3^2\right)\tau^3 \\ - \left(\tfrac{1}{18}c_2^3c_3 + \tfrac{1}{3}c_2c_3^2\right)\tau^4 - \left(\tfrac{3}{70}c_2^2c_3^2 + \tfrac{1}{14}c_3^3\right)\tau^5 - \tfrac{1}{60}c_2c_3^3\tau^6 - \tfrac{1}{378}c_3^4\tau^7.$$

Note that the homogeneous part of the equation is the same as for the next-to-leading-order equation. Consequently the constants c_4, c_5 and c_6 may be set at zero. The process may be repeated as often as one wishes to establish the Laurent series to the desired order.

In the instance of the Left Painlevé Series indicated by the third set of resonances the situation is similar. The point of listing these results is to demonstrate that the Laurent series, be they Left or Right Painlevé Series, may be constructed by successive applications of the idea of partial solution plus a function to be determined. To determine this function we linearize the equation thereby making it an approximation[11].

We turn now to the second of the possible patterns of resonances. The existence of both positive and negative resonances (apart from the 'generic' -1) has been something of a bugbear for workers in the area of singularity analysis even after the case of negative resonances had been correctly interpreted in the seminal work of Feix *et al* [24]. Equation (7) provides an eloquent vehicle for explanation since it is integrable in closed form and has the explicit solution

$$x = \frac{1}{t - t_0} + \frac{1}{t - t_1} + \frac{1}{t - t_2}, \tag{8}$$

where t_0, t_1 and t_2 are the required three constants of integration and indicate that the general solution has three movable singularities. By the implication of the notation we are looking for a solution expanded about the singularity at t_0 and we assume for the sake of a simple discussion that $|t_1 - t_0| < |t_2 - t_0|$. Then the solution for the first set of resonances belongs to the punctured disc centred on t_0 of radius $|t_1 - t_0|$ and that the third belongs to the complex t-plane external to the disc defined by $|t| > |t_2 - t_0|$. Evidently the middle set of resonances relates to the solution in the annulus[12] defined by $|t_1 - t_0| < |t| < |t_2 - t_0|$. In the 'next-to' approach used for the first and third set of resonances we dropped a term in the solution for $g(\tau)$ which had no place in the series. If one has both positive and negative resonances, the exclusion of terms is not possible. Proceeding in the manner established above we have

$$\dddot{g} + 8\tau^{-1}\ddot{g} + 12\tau^{-2}\dot{g} = 0$$

with the solution

$$g = c_1\tau^{-3} + c_2\tau^{-2} + c_3$$

[10] In a careful application of this method one would also not rely upon the last term, but here we simply wish to illustrate how the method works rather than obtain a precise result.

[11] One should note that for the Left Painlevé Series the term, τ^{-2}, is included although it properly belongs to the generic resonance of -1. The usual argument regarding the meaning of the resonance at -1 clearly shows that the constant associated with the location of the movable singularity and c_3 are conflated.

[12] For a detailed discussion, see [4].

for the next-to-leading-order behaviour. In the second iteration we obtain a third-order equation for g with solution

$$\begin{aligned} g = {} & \frac{1}{30\,618}c_1^4\tau^{-9} + \frac{1}{3240}c_1^3c_2\tau^{-8} + \left(\frac{1}{840}c_1^2c_2^2 + \frac{1}{378}c_1^3\right)\tau^{-7} \\ & + \left(\frac{1}{432}c_1c_2^3 - \frac{1}{486}c_1^3c_3 + \frac{1}{54}c_1^2c_2\right)\tau^{-6} \\ & \times \left(\frac{1}{18}c_1^2 - \frac{1}{45}c_1^2c_2c_3 + \frac{1}{480}c_2^4 + \frac{1}{20}c_1c_2^2\right)\tau^{-5} \\ & + \left(\frac{1}{16}c_2^3 + \frac{1}{4}c_1c_2 - \frac{1}{8}c_1c_2^2c_3 - \frac{1}{6}c_1^2c_3\right)\tau^{-4} \\ & + \left(\frac{2}{9}c_2^3c_3 + \frac{8}{9}c_1c_2c_3 - \frac{1}{3}c_4 - \frac{8}{27}c_1^2c_3^2\right)\tau^{-3} - \frac{1}{2}\left(c_5 + c_1c_2c_3^2\right)\tau^{-2} \\ & + \left(\frac{3}{4}c_2^2c_3^2 - c_1c_3^2\right)\tau^{-1} + c_6 + \left(\frac{1}{6}c_2c_3^3 - c_3^2\right)\tau - \frac{1}{5}c_3^3\tau^2 - \frac{1}{90}c_3^4\tau^3 \\ & + \left[\left(\frac{2}{3}c_1c_2 + \frac{1}{6}c_2^3 - \frac{2}{9}c_1^2c_3\right)c_3\tau^{-3} + c_1c_2c_3^2\tau^{-2} + \left(c_2 + \frac{2}{9}c_1c_3\right)c_3^2\right]\log\tau. \end{aligned}$$

Note that in this case one does not have the choice of dropping one of the terms from the solution to the equation for the next-to-leading-order correction, since the exponents in the Laurent expansion both increase and decrease from the leading order. The consequence of this is very much evident in the next-to-next-to-leading-order correction in which logarithmic terms appear. These terms are spurious as is quite evident from the explicit solution given in (8).

We now turn to system (1). We make the substitutions

$$u = \alpha\tau^{-1} + f(\tau) \qquad v = \beta\tau^{-1} + g(\tau) \qquad w = \gamma\tau^{-1} + h(\tau) \tag{9}$$

and, after we replace α, β and γ by their common values i (*mutatis mutandis* the result for $-$i is the same), we obtain the system

$$\begin{aligned} & 2\tau^{-3}f + 2\tau^{-2}\dot{f} + \tau^{-1}\ddot{f} = 4\tau^{-3}f \\ & 2\tau^{-3}g + 2\tau^{-2}\dot{g} + \tau^{-1}\ddot{g} = 4\tau^{-3}g \\ & 2\tau^{-3}h + 2\tau^{-2}\dot{h} + \tau^{-1}\ddot{h} = 4\tau^{-3}h \end{aligned} \tag{10}$$

which has the solution

$$f = c_1\tau + c_2\tau^{-2} \qquad g = c_3\tau + c_4\tau^{-2} \qquad h = c_5\tau + c_6\tau^{-2}. \tag{11}$$

As Contopoulos *et al* [16] observed, no logarithmic term enters at the triple -1 resonance. Furthermore three arbitrary constants enter at this resonance. This does not mean that there are seven arbitrary constants. The location of the singularity is merged with the constants c_2, c_4 and c_6. As we have observed above, the constraint, (2), reduces the number of free parameters to five.

Formerly this case had been treated as if it were a Right Painlevé Series, i.e. the contribution of the τ^{-2} terms was ignored. However, motivated by the integrable example of (7) we may consider (11) as an indication of the existence of a Laurent expansion valid in an annulus centred on the singularity as in the middle case for (7). When we calculate the next-to-next-to-leading-order behaviour by making the obvious substitution using (11), we obtain the f, g, h functions to be some extremely lengthy and complicated expressions. We do note the well-awaited and necessary intrusion of logarithmic terms as in the case of (7) for mixed resonances.

For equation (7) it was evident that the occurrence of logarithmic terms was quite spurious. Here we cannot be so dogmatic since we do not know the solution of system (1). However, through (11) we see that the requisite number of arbitrary constants has already entered the expansion without any need for the introduction of logarithmic terms. Consequently one would expect the substitution of a series of the form $\sum_{i=-\infty}^{\infty} a_i\tau^i$ for each of u, v and w to be quite consistent. A phenomenon such as we have observed above may have been behind the comment of Contopoulos *et al* [17] about the interaction[13] of the resonances at -1 and 2 causing the introduction of logarithmic terms in their analysis.

5. Conclusion

We have presented a singularity analysis of system (1) subject to the constraint (2) which are the vacuum equations for the Bianchi IX model. In this form of the model the energy density, pressure and cosmological constant are zero. The time scale we used involves a logarithmic transformation. An alternative representation is found for example in [18]. One of the problems when analyzing physical systems is that although the theory, general relativity, is coordinate independent, the actual calculations become coordinate dependent. However, it is important to note that the variables used in the analysis do not introduce a mathematical obstacle to integrability in the natural variables of the physical problem.

In the standard singularity analysis of (1) we obtained, up to cyclic triplication, two distinct cases. In case 1 the leading-order exponents are $(-1, 1, 1)$ with coefficients $(\pm \mathrm{i}, 1/\beta, 1/\gamma)$, where β and γ are arbitrary and resonances $-1, 0(2), 1(2), 2$. On the other hand, for case 3, the corresponding results are $(-1, -1, -1), (\pm \mathrm{i}, \pm \mathrm{i}, \pm \mathrm{i})$, $s = -1(3), 2(3)$. Both cases contain six arbitrary constants (reduced to five when (2) is taken into account) in their Laurent expansions and are consistent.

Earlier papers [16, 20, 30] reported only four constants for case 3. This was due to an incomplete analysis of the triple -1 resonance. When we treated this resonance as one would treat a triple root of any other value, we recovered a full set of constants. Unlike Latifi *et al* [30], who expressed the solution as two separate series, one a Left Painlevé Series and one a Right Painlevé Series thereby neatly bypassing the question of introducing logarithmic terms, we interpret this case as a Laurent series valid in an annulus [4] so that both negative and positive powers do not enter their respective singularities. This comes in accordance with the density of the singularities located in the complex plane as was demonstrated numerically in [13]. Consequently this suggests that these Laurent series reflect local integrability.

If we extend our analysis to include nonintegral rational exponents we find that there is no possibility of terms balancing.

As we indicated in the introduction our purpose is not to prove the integrability or nonintegrability of the Mixmaster Universe but to present a complete singularity analysis without unsustainable inferences. Since the domain of validity of the series is usually unknown and the proving of convergence of such complex series is generally not feasible, one does well to examine other systems which exhibit similar behaviour in their singularity properties, but their solution is explicitly known and make a direct comparison. It is an easy matter to construct a system of three second-order nonlinear ordinary differential equations which has a triple -1 resonance (amongst other negative and positive values) and an explicit solution completely devoid of logarithms [7]. Consequently the existence of such features is not inconsistent with integrability. From our singularity analysis we cannot conclude that there is any evidence which points towards the integrability or nonintegrability of the Mixmaster

[13] Here without apostrophes.

Universe. What we can conclude is that the singularity analysis of the Mixmaster Universe is a delicate matter and not a subject for hasty conclusions. It is unfortunate that the singularity analysis is inconclusive for one then has to rely upon numerical experimentation which is a tedious and expensive affair—like most affairs.

Acknowledgments

We thank one of the referees for drawing our attention to references [19, 21]. KA thanks the State (Hellenic) Scholarship Foundation and the University of KwaZulu-Natal for their support. PGLL thanks the University of the Aegean for the provision of facilities while this work was being prepared and the University of KwaZulu-Natal for its continued support.

References

[1] Ablowitz M J, Ramani A and Segur H 1978 Nonlinear evolution equations and ordinary differential equations of Painlevé type *Lett. Nuovo Cimento* **23** 333–7
[2] Ablowitz M J, Ramani A and Segur H 1980 A connection between nonlinear evolution equations and ordinary differential equations of P type I *J. Math. Phys.* **21** 715–21
[3] Ablowitz M J, Ramani A and Segur H 1980 A connection between nonlinear evolution equations and ordinary differential equations of P type II *J. Math. Phys.* **21** 1006–15
[4] Andriopoulos K and Leach P G L 2006 An interpretation of the presence of both positive and negative nongeneric resonances in the singularity analysis *Phys. Lett.* A **359** 199–203
[5] Andriopoulos K, Leach P G L and Maharaj A 2007 On differential sequences *Preprint* arXiv:0704.3243
[6] Andriopoulos K and Leach P G L 2008 Analysis of an FRW cosmological model *Central Eur. J. Phys.* (to appear)
[7] Andriopoulos K and Leach P G L 2007 The occurence of a triple -1 resonance in the standard singularity analysis (preprint DICSE, University of the Aegean, Samos, Greece)
[8] Belinskii V A and Khalatnikov I M 1969 On the nature of the singularities in the general solution of the gravitational equations *Sov. Phys.—JETP* **29** 911–7
[9] Belinskii V A and Khalatnikov I M 1970 General solution of the gravitational equations with a physical singularity *Sov. Phys.—JETP* **30** 1174–80
[10] Belinskii V A, Khalatnikov I M and Lifshitz E M 1970 Oscillatory approach to the singular point in relativistic cosmology *Adv. Phys.* **19** 525–73
[11] Belinskii V A, Gibbons G W, Page D N and Pope C N 1978 Asymptotically Euclidean Bianchi IX in quantum gravity *Phys. Lett.* **79B** 433–5
[12] Bogoyavlenski O I 1985 *Methods in the Qualitative Theory of Dynamical Systems in Astrophysics and Gas Dynamics* (Berlin: Springer)
[13] Bountis T C and Drossos L B 1997 Evidence of a natural boundary and nonintegrability of the Mixmaster Universe model *J. Nonlinear Sci.* **7** 45–55
[14] Christiansen F, Rugh H H and Rugh S E 1995 Nonintegrability of the mixmaster universe *J. Phys. A: Math. Gen.* **28** 657–67
[15] Conte R, Fordy A P and Pickering A 1991 A perturbative Painlevé approach to nonlinear differential equations *Physica* **69D** 33–58
[16] Contopoulos G, Grammaticos B and Ramani A 1993 Painlevé analysis for the mixmaster universe model *J. Phys. A: Math. Gen.* **25** 5795–9
[17] Contopoulos G, Grammaticos B and Ramani A 1994 The mixmaster universe model, revisited *J. Phys. A: Math. Gen.* **27** 5357–61
[18] Contopoulos G, Grammaticos B and Ramani A 1995 The last remake of the mixmaster universe model *J. Phys. A: Math. Gen.* **28** 5313–22
[19] Cornish Neil J and Levin Janna J 1997 Mixmaster Universe: a chaotic fairy tale *Phys. Rev.* D **55** 7489–510
[20] Cotsakis S and Leach P G L 1994 Painlevé analysis of the Mixmaster Universe *J. Phys. A: Math. Gen.* **27** 1625–31
[21] Creighton T D and Hobill D W 1994 *Deterministic Chaos in General Relativity* ed D W Hobill, A Burd and A Coley (New York: Plenum)

[22] Demaret J and Scheen C 1996 Painlevé singularity analysis of the perfect fluid Bianchi type-iX relativistic cosmological model *J. Phys. A: Math. Gen.* **29** 59–76

[23] Euler M, Euler N and Leach P G L 2007 The Riccati and Ermakov-pinney hierarchies *J. Nonlinear Math. Phys.* **14** 290–302

[24] Feix M R, Géronimi C, Cairó L, Leach P G L, Lemmer R L and Bouquet S É 1997 On the singularity analysis of ordinary differential equations invariant under time translation and rescaling *J. Phys. A: Math. Gen.* **30** 7437–61

[25] Fordy A P and Pickering A 1991 Analysing negative resonances in the Painlevé test *Phys. Lett.* **160A** 347–54

[26] Géronimi C, Leach P G L and Feix M R 2002 Singularity analysis and a function unifying the Painlevé and Ψ series *J. Nonlinear Math. Phys.* **9 s-2** 36–48

[27] Hobill D, Bernstein D and Welge M 1991 The mixmaster universe as a dynamical system *Class. Quantum Grav.* **8** 1155–71

[28] Kowalevski S 1889 Sur la problème de la rotation d'un corps solide autour d'un point fixe *Acta Math.* **12** 177–232

[29] Kruskal M D 1992 Flexibility in applying the Painlevé Test *Painlevé Transcendents, their Asymptotics and Physical Applications (NATO ASI Series B: Physics* vol 278*)* ed D Levi and P Winternitz (New York: Plenum)

[30] Latifi A, Musette M and Conte R 1994 The Bianchi IX (Mixmaster) cosmological model is not integrable *Phys. Lett.* **194A** 83–92

[31] Leach P G L, Govinder K S, Brazier T I and Schei C S 1993 The ubiquitous time-dependent simple harmonic oscillator *South Afr. J. Sci.* **89** 126–30

[32] Lemmer R L and Leach P G L 1993 The Painlevé test, hidden symmetries and the equation $y'' + yy' + ky^3 = 0$ *J. Phys. A: Math. Gen.* **26** 5017–24

[33] Misner C W 1969 Mixmaster universe *Phys. Rev. Lett.* **22** 1071–4

[34] Misner C W 1969 Quantum cosmology I *Phys. Rev.* **186** 1319–27

[35] Misner C W 1969 Absolute zero of time *Phys. Rev.* **186** 1328–33

[36] Richard A C 1994 *Painlevé Analysis and Partial Integrability of Some Dynamical Systems* (Dissertation: University of Natal, Durban, Republic of South Africa)

[37] Springael J, Conte R and Musette M 1998 On the exact solutions of the Bianchi IX cosmological model in the proper time *Regular and Chaotic Dynamics* **3** 3–8

[38] Wainwright J and Hsu L 1989 A dynamical systems approach to Bianchi cosmologies: Orthogonal models of Class A *Class. Quantum Grav.* **6** 1409–31

[39] Ziglin S L 1983 Branching of solutions and non-existence of first integrals in Hamiltonian mechanics I *Funct. Anal. Appl.* **16** 181–9

[40] Ziglin S L 1983 Branching of solutions and non-existence of first integrals in Hamiltonian mechanics II *Funct. Anal. Appl.* **17** 6–17